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Contents

AN ADELIC EXTENSION OF THE JONES POLYNOMIAL

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Abstract. In this paper we represent the classical braids in the classical and the adelic Yokonuma–Hecke algebras. More precisely, we define the completion of the framed braid group and we introduce the adelic Yokonuma–Hecke algebras, in analogy to the notions of p-adic framed braids and p-adic Yokonuma–Hecke algebras introduced in [3, 4]. We further construct an adelic Markov trace, analogous to a p-adic Markov trace constructed in [4]. Using the traces in [2] and the adelic Markov trace we define topological invariants of classical knots and links, upon imposing some condition (in analogy to the invariants of framed links defined in [4]). These invariants are related to a cubic skein relation coming from the Yokonuma–Hecke algebra.

1. Introduction

Let B_n be the classical braid group on n strands. As usual we denote $\sigma_1, \ldots, \sigma_{n-1}$ the elementary braids which generate B_n under the defining relations:

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} \qquad \text{and} \qquad \sigma_i\sigma_j = \sigma_j\sigma_i \quad \text{for} \quad |i-j| > 1.$$

From the topological point of view, an element in B_n consists in n arcs embedded in a thickened square, such that the ends are arranged into n collinear top endpoints and into n collinear bottom endpoints and such that there are no local maxima or minima. The braid generator σ_i is a positive crossing between the ith and the $(i+1)$st strand, while σ_i^{-1} is the opposite crossing. The operation in B_n corresponds to the concatenation of two braids and the braid relations reflect allowed topological moves.

Closing a braid β means to join with simple arcs the corresponding top and bottom endpoints of β, and it gives rise to an oriented knot or link, denoted $\widehat{\beta}$. Conversely, by the classical Alexander theorem, an oriented knot or link can be isotoped to the closure of a braid. *Isotopy* is the notion of topological equivalence for knots and links. Further, by the classical Markov theorem, isotopy classes of oriented knots or links are in bijective correspondence with equivalence classes of braids. More precisely, the natural inclusions $B_n \subset B_{n+1}$, induced by adding at the end of the braid an extra identity strand, give rise to the direct limit B_∞. Then we have the following classical result.

Theorem 1 (Markov, 1935)**.** *Isotopy classes of oriented links are in bijection with equivalence classes of braids in B_∞, where the equivalence relation is generated by the two moves:*

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(i) Conjugation: $\alpha\beta \sim \beta\alpha, \quad \alpha, \beta \in B_n$

(ii) Markov move: $\alpha \sim \alpha{\sigma_n}^{\pm 1}, \quad \alpha \in B_n$

Using the theory of braids and Ocneanu's Markov trace on the Iwahori–Hecke algebras of type A (see [1]), V.F.R. Jones constructed in [1] the 2-variable Jones polynomial, an isotopy invariant of oriented classical knots and links. The Iwahori–Hecke algebras of type A can be described naturally as quotients of the group algebras $\mathbb{C}B_n$ over a quadratic relation.

The Yokonuma–Hecke algebras $\mathrm{Y}_{d,n}(u)$ are similar algebraic objects and have a natural topological interpretation as quotients of the modular framed braid group algebras $\mathbb{C}\mathcal{F}_{d,n}$ (classical framed braids with framings modulo d) over certain quadratic relations. d is any non–negative integer, and if $d = 1$ then the algebra $\mathrm{Y}_{1,n}(u)$ is isomorphic to the above-mentioned Iwahori–Hecke algebra. Topologically, $d = 1$ means all framings zero, so $\mathrm{Y}_{1,n}(u)$ is really related to classical braids (with no framings). In [2] Markov traces are constructed on the Yokonuma–Hecke algebras. For $d = 1$ this trace coincides with the Ocneanu trace.

In [3] we introduced the p-adic framed braids and the p-adic Yokonuma–Hecke algebras. Further, in [4] we constructed a p-adic Markov trace, which we used, together with the traces in [2], in order to construct an infinite family of topological invariants of framed links, upon imposing some condition. In the present paper we relate the Yokonuma–Hecke algebras, for $d \neq 0$, to classical knots and links via a homomorphism of the classical braid group B_n. Further, using the Markov traces in [2] we construct an infinite family of isotopy invariants of classical knots and links, upon imposing some condition. We further define the completion of the framed braid group and we introduce the adelic Yokonuma–Hecke algebras, into which the classical braid group also maps homomorphically. We also construct, in analogy to [4], an adelic Markov trace, which we use for constructing an isotopy invariant of classical knots and links, the adelic extension of the 2–variable Jones polynomial. Our invariants satisfy a cubic skein relation coming from the Yokonuma–Hecke algebras. In an effort to keep this paper light we omit some technical details, which are mostly to be found in [4].

The Yokonuma–Hecke algebras are very versatile algebraic objects, in the sense that they can be used for completely different topological interpretations. They comprise the only examples we know of having this property. Indeed, apart from the framed braids and the classical braids, they are also related to singular braids, as there is a monoid representation of the singular braid monoid algebra to $\mathrm{Y}_{d,n}(u)$ (not onto). Then the traces in [2] are also Markov traces on the singular braid monoid. See [5] for details.

2. An adelic representation of the braid group

2.1. *The Yokonuma–Hecke algebra.* Fix $u \in \mathbb{C}\backslash\{0, 1\}$. Given two positive integers d and n, we denote $\mathrm{Y}_{d,n} = \mathrm{Y}_{d,n}(u)$ the Yokonuma–Hecke algebra, which is a unital associative algebra over $\mathbb{C}$, defined by the generators:

$$1, g_1, \ldots, g_{n-1}, t_1, \ldots, t_n$$

subject to the following relations:

$$
\begin{array}{rcll}
g_i g_j & = & g_j g_i & \text{for } |i-j|>1 \\
g_i g_j g_i & = & g_j g_i g_j & \text{for } |i-j|=1 \\
t_i t_j & = & t_j t_i & \text{for all } i,j \\
t_j g_i & = & g_i t_{s_i(j)} & \text{for all } i,j \\
t_j^d & = & 1 & \text{for all } j
\end{array} \tag{1}
$$

where $s_i(j)$ is the result of applying the transposition $s_i = (i, i+1)$ to j, together with the extra quadratic relations:

$$g_i^2 = 1 + (u-1)\, e_{d,i} - (u-1)\, e_{d,i}\, g_i \qquad \text{for all } i \tag{2}$$

where

$$e_{d,i} := \frac{1}{d}\sum_{m=0}^{d-1} t_i^m t_{i+1}^{-m}. \tag{3}$$

Remark 1. For all $1 \le i \le n$, let $C_{d,i} = \{1, t_i, t_i^2, \ldots, t_i^{d-1}\}$ denote the cyclic group containing all possible framings of the ith strand of a modular framed braid. Notice that $C_{d,i} \simeq \mathbb{Z}/d\mathbb{Z}$ for all i. We define also the group $H := C_{d,1} \times C_{d,2} \times \ldots \times C_{d,n} \simeq (\mathbb{Z}/d\mathbb{Z})^n$. From the defining relations among the t_i's we deduce that the groups $C_{d,i}$ and H can be regarded inside $\mathrm{Y}_{d,n}$.

The Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}$ is a quotient of the modular framed braid group algebra $\mathbb{C}\mathcal{F}_{d,n}$ over the quadratic relations (2). The modular framed braid group $\mathcal{F}_{d,n}$ is generated by the braiding generators σ_i and the framing generators $t_1, \ldots, t_n$, where t_j means the identity braid with framing one on the jth strand and framing zero on the other strands. Corresponding the braiding generators σ_i to the algebra generators g_i, relations (1) furnish a presentation for $\mathcal{F}_{d,n}$. Elements of $\mathcal{F}_{d,n}$ are classical framed braids, but with framings modulo d. The elements $e_{d,i}$ are in the algebra $\mathbb{C}\mathcal{F}_{d,n}$ as well as in the quotient algebra $\mathrm{Y}_{d,n}$. They are expressions of the framing generators t_i, t_{i+1} and it is easy to check that they are idempotents.

The Yokonuma–Hecke algebra was originally introduced by T. Yokonuma [9].

2.2. *The adelic Yokonuma–Hecke algebra.* Let us denote by $\mathbb{N}$ the set of positive integers regarded as a directed set with the usual order. We shall denote by $\mathbb{N}^{\sim}$ the directed set of positive integers regarded with respect to the partial order defined by the divisibility relation. The notation $d|d'$ means d divides d'.

For d, $d' \in \mathbb{N}$ with $d|d'$ we consider the natural connecting ring homomorphism $\rho_d^{d'}$, defined in [4], (1.17):

$$\rho_d^{d'} : \mathrm{Y}_{d',n} \longrightarrow \mathrm{Y}_{d,n} \tag{4}$$

More precisely, we denote $\vartheta_d^{d'}$ the natural epimorphism:

$$
\begin{array}{rccc}
\vartheta_d^{d'} : & \mathbb{Z}/d'\mathbb{Z} & \longrightarrow & \mathbb{Z}/d\mathbb{Z} \\
& m & \mapsto & m \,(\mathrm{mod}\, d)
\end{array} \tag{5}
$$

The inverse limit $\widehat{\mathbb{Z}}$ of the inverse system of groups $(\mathbb{Z}/d\mathbb{Z}, \vartheta_d^{d'})$ indexed by $\mathbb{N}^\sim$ is called *the completion of* $\mathbb{Z}$:

$$\widehat{\mathbb{Z}} = \varprojlim_{d\in\mathbb{N}^\sim} \mathbb{Z}/d\mathbb{Z}$$

Our references for inverse limits are mainly [7] and [8].

By componentwise multiplication, epimorphism (5) defines the epimorphism:

$$\varpi_d^{d'} : \ (\mathbb{Z}/d'\mathbb{Z})^n \ \longrightarrow \ (\mathbb{Z}/d\mathbb{Z})^n \tag{6}$$

Extension to the B_n–part by the identity map yields the epimorphism:

$$\varpi_d^{d'} \cdot \mathrm{id} : \ \mathcal{F}_{d',n} \ \longrightarrow \ \mathcal{F}_{d,n} \tag{7}$$

Definition 1. The *completion* $\mathcal{F}_{\infty,n}$ *of the framed braid group* $\mathcal{F}_n$ is defined as the inverse limit of the inverse system of groups $(\mathcal{F}_{d,n},\, \varpi_d^{d'} \cdot \mathrm{id})$:

$$\mathcal{F}_{\infty,n} := \varprojlim_{d\in\mathbb{N}^\sim} \mathcal{F}_{d,n}$$

The linear extension of map (7) yields an algebra epimorphism:

$$\varrho_d^{d'} : \mathbb{C}\mathcal{F}_{d',n} \longrightarrow \mathbb{C}\mathcal{F}_{d,n} \tag{8}$$

Remark 2. The braid group B_n acts on $\widehat{\mathbb{Z}}^n$ by permuting the factors, so we may consider the group $\widehat{\mathbb{Z}}^n \rtimes B_n$. It is easy to construct an isomorphism between the groups $\widehat{\mathbb{Z}}^n \rtimes B_n$ and $\mathcal{F}_{\infty,n}$ (proof analogous to Theorem 1 in [3]). We note, though, that this isomorphism does not carry through on the level of the algebras $\mathbb{C}(\widehat{\mathbb{Z}}^n \rtimes B_n)$ and $\varprojlim_{d\in\mathbb{N}^\sim} \mathbb{C}\mathcal{F}_{d,n}$ (see [4] for more details).

Passing now to the quotient algebras by relations (2) we obtain the following algebra epimorphism:

$$\rho_d^{d'} : \mathrm{Y}_{d',n} \longrightarrow \mathrm{Y}_{d,n} \tag{9}$$

Definition 2. The *adelic Yokonuma–Hecke* algebra $\mathrm{Y}_{\infty,n}(u) = \mathrm{Y}_{\infty,n}$ is defined as the inverse limit of the inverse system of rings $(\mathrm{Y}_{d,n}, \rho_d^{d'})$ indexed by $\mathbb{N}^\sim$:

$$\mathrm{Y}_{\infty,n} = \varprojlim_{d\in\mathbb{N}^\sim} \mathrm{Y}_{d,n}$$

Hence, elements in $\mathrm{Y}_{\infty,n}$ are infinite sequences of elements in the algebras $\mathrm{Y}_{d,n}$, for $d \in \mathbb{N}^\sim$, which are coherent in the sense of maps (5) – (9). Moreover, the definition of the connecting maps $\rho_d^{d'}$ do not involve the elements g_i, so we shall denote also by g_i the elements in $\mathrm{Y}_{\infty,n}$ corresponding to the infinite constant sequence (g_i).

For all $0 \leq i \leq n-1$, define now the groups $H_{d,i}$ as follows:

$$H_{d,i} = \{1, t_i t_{i+1}^{-1}, t_i^2 t_{i+1}^{-2}, \ldots, t_i^{d-1} t_{i+1}\}$$

Then, the element $e_{d,i}$ is the average of the elements of the group $H_{d,i}$:

$$e_{d,i} = \frac{1}{d} \sum_{x\in H_{d,i}} x$$

Then $\rho_d^{d'}(H_{d',i}) = H_{d,i}$ for all $d|d'$. Hence, we deduce the following result.

Lemma 1. *For all i and for d, d' such that $d|d'$, we have:*

$$\rho_d^{d'}(e_{d',i}) = e_{d,i}.$$

We shall denote by e_i the sequence $(e_{d,i})_{d\in\mathbb{N}^\sim}$ in $\mathrm{Y}_{\infty,n}$:

$$e_i = (e_{d,i})_{d\in\mathbb{N}^\sim} \tag{10}$$

Proposition 1. *For all i the following relations hold in $\mathrm{Y}_{\infty,n}$:*

(1) $e_i g_i = g_i e_i$
(2) $g_i^2 = 1 + (u-1)e_i g_i - (u-1)g_i$.

2.3. *Representing the braid group.* The defining relations of $\mathrm{Y}_{d,n}$ imply that the map:

$$\begin{array}{rccc} \flat_{d,n}: & B_n & \longrightarrow & \mathrm{Y}_{d,n} \\ & \sigma_i & \mapsto & g_i \end{array} \tag{11}$$

defines a representation of B_n in $\mathrm{Y}_{d,n}$. Under this representation the generators g_i of the algebra $\mathrm{Y}_{d,n}$ correspond to the braid crossings σ_i. The generators t_j, though, loose their topological interpretation as framing generators and they are just considered algebraically.

Further, for all d, d', d'' such that $d|d'$ and $d'|d''$ we have the following commutative diagram:

$$\begin{tikzcd} \cdots & B_n \arrow[l] \arrow[d, "\flat_{d,n}"'] & B_n \arrow[l, "\mathrm{Id}"'] \arrow[d, "\flat_{d',n}"'] & B_n \arrow[l, "\mathrm{Id}"'] \arrow[d, "\flat_{d'',n}"'] & \cdots \arrow[l] \\ \cdots & \mathrm{Y}_{d,n} \arrow[l] & \mathrm{Y}_{d',n} \arrow[l, "\rho_d^{d'}"'] & \mathrm{Y}_{d'',n} \arrow[l, "\rho_{d'}^{d''}"'] & \cdots \arrow[l] \end{tikzcd} \tag{12}$$

By taking inverse limits in the above diagram we obtain the following representation of the classical braid group in the adelic Yokonuma–Hecke algebra:

$$\flat_{\infty,n}: B_n \longrightarrow \mathrm{Y}_{\infty,n} \tag{13}$$

where:

$$\flat_{\infty,n} := \varprojlim_{d\in\mathbb{N}^\sim} \flat_{d,n}$$

3. An adelic Markov trace

3.1. *A modular Markov trace.* It is known that the Yokonuma–Hecke algebra supports a Markov trace [2]. More precisely, for fixed d we consider the inductive system $(\mathrm{Y}_{d,n})_{n\in\mathbb{N}}$ associated to the natural inclusion $\mathrm{Y}_{d,n} \subset \mathrm{Y}_{d,n+1}$, for all $n \in \mathbb{N}$. Let $\mathrm{Y}_{d,\infty}$ be the corresponding inductive limit. In [2] the following theorem is proved.

Theorem 2 (Juyumaya, 2004)**.** *Let z, x_1, …, $x_{d-1} \in \mathbb{C}$ and let d be a positive integer. For all $n \in \mathbb{N}$ there exists a unique linear map $\mathrm{tr}_d = (\mathrm{tr}_{d,n})_{n\in\mathbb{N}}$:*

$$\mathrm{tr}_d : \mathrm{Y}_{d,\infty} \longrightarrow \mathbb{C}$$

satisfying the rules:

$$\begin{array}{rcll} \mathrm{tr}_{d,n}(ab) & = & \mathrm{tr}_{d,n}(ba) & \\ \mathrm{tr}_{d,n}(1) & = & 1 & \\ \mathrm{tr}_{d,n+1}(ag_n) & = & z\,\mathrm{tr}_{d,n}(a) & (a \in \mathrm{Y}_{d,n}) \\ \mathrm{tr}_{d,n+1}(at_{n+1}^m) & = & x_m \mathrm{tr}_{d,n}(a) & (a \in \mathrm{Y}_{d,n}\,,\ 1 \leq m \leq d-1). \end{array}$$

The proof of Theorem 2 rests on the fact that the algebra $\mathrm{Y}_{d,n+1}$ admits an inductive linear basis, where either g_n or t_{n+1}^m appears at most once. Note that, for $d = 1$ the trace restricts to the first three rules and it coincides with Ocneanu's trace on the Iwahori–Hecke algebra, which was used in [1] to construct the 2–variable Jones polynomial for classical knots and links.

3.2. Let R be the polynomial ring $\mathbb{C}[z]$ and let $R[X_d]$ be the polynomial ring with coefficients in R and variables x_a, where $a \in \mathbb{Z}/d\mathbb{Z}$. Let also $d|d'$. The natural map $x_a \mapsto x_b$ where $b := \vartheta_d^{d'}(a)$ (recall (5)), defines a ring epimorphism:

(14) $$\xi_d^{d'} : R\,[X_{d'}] \longrightarrow R\,[X_d]$$

We now have the following result (compare with Lemma 7[4]).

Lemma 2. *The family* $\left(R[X_d], \xi_d^{d'}\right)$ *indexed by* $\mathbb{N}^{\sim}$*, is an inverse system.*

We shall then consider the inverse limit:

$$\varprojlim_{d \in \mathbb{N}^{\sim}} R[X_d]$$

Notice that $\varprojlim_{d\in\mathbb{N}^{\sim}} R[X_d]$ can be regarded as the polynomial ring over $\mathbb{C}$ in the variables z and x_α, where $\alpha \in \widehat{\mathbb{Z}}$. The ring $\varprojlim_{d\in\mathbb{N}^{\sim}} R[X_d]$ turns out to be an integral domain.

Now, for all $n \in \mathbb{N}$ and for all d, d', d'' such that $d|d'$ and $d'|d''$, we have the following commutative diagram (compare with Lemma 6[4]):

(15) $$\begin{tikzcd} \cdots & \mathrm{Y}_{d,n} \arrow[l] \arrow[d, "\mathrm{tr}_{d,n}"'] & \mathrm{Y}_{d',n} \arrow[l, "\rho_d^{d'}"'] \arrow[d, "\mathrm{tr}_{d',n}"'] & \mathrm{Y}_{d'',n} \arrow[l, "\rho_{d'}^{d''}"'] \arrow[d, "\mathrm{tr}_{d'',n}"'] & \cdots \arrow[l] \\ \cdots & R\,[X_d] \arrow[l] & R\,[X_{d'}] \arrow[l, "\xi_d^{d'}"'] & R\,[X_{d''}] \arrow[l, "\xi_{d'}^{d''}"'] & \cdots \arrow[l] \end{tikzcd}$$

Finally, we note that there are natural inclusions $\mathrm{Y}_{\infty,n} \subset \mathrm{Y}_{\infty,n+1}$, for all $n \in \mathbb{N}$. Let

$$\mathrm{Y}_\infty := \varinjlim_{n\in\mathbb{N}} \mathrm{Y}_{\infty,n}$$

the associated inductive limit. We then have the following.

Theorem 3. *There exists a unique linear Markov trace* $\mathrm{tr}_\infty = (\mathrm{tr}_{\infty,n})_{n\in\mathbb{N}}$,

$$\mathrm{tr}_\infty : \mathrm{Y}_\infty \longrightarrow \varprojlim_{d\in\mathbb{N}^{\sim}} R[X_d]$$

such that

$$\begin{array}{rcl} \mathrm{tr}_{\infty,n}(ab) & = & \mathrm{tr}_{\infty,n}(ba) \\ \mathrm{tr}_{\infty,n}(1) & = & 1 \\ \mathrm{tr}_{\infty,n+1}(ag_n) & = & z\,\mathrm{tr}_{\infty,n}(a) \\ \mathrm{tr}_{\infty,n+1}(ay_{n+1}) & = & \mathrm{tr}_{\infty,n}(a)\mathrm{tr}_{\infty,n+1}(y) \end{array}$$

where a, $b \in \mathrm{Y}_{\infty,n}$ *and* y_{n+1} *is the element in* $\widehat{\mathbb{Z}}^{n+1}$ *with* $y \in \widehat{\mathbb{Z}}$ *in the position* $n+1$ *and* 0 *otherwise, that is:* $y_{n+1} = (0, \ldots, 0, y)$.

Proof. It follows immediately from the commutative diagram (15) and from the existence and uniqueness of the traces tr_d. □

4. The E–condition

4.1. The representations (11) and (13) of the braid group through the classical and the adelic Yokonuma–Hecke algebras, composed with the Markov traces tr_d and tr_∞ of Theorems 2 and 3, map braids to complex polynomials. In view of the Alexander and Markov topological theorems we would like to construct isotopy invariants for classical oriented knots and links. According to Theorem 1, such an invariant has to agree on the links $\widehat{\alpha}$, $\widehat{\alpha\sigma_n}$ and $\widehat{\alpha\sigma_n^{-1}}$, for any $\alpha \in B_n$. Following Jones' construction of the 2-variable Jones polynomial for classical knots [1], we will try to define knot isotopy invariants by re-scaling and normalizing the traces tr_d and the adelic trace tr_∞. By the equation:

$$g_i^{-1} = g_i - (u^{-1} - 1)\, e_{d,i} + (u^{-1} - 1)\, e_{d,i}\, g_i \tag{16}$$

we have:

$$\mathrm{tr}_d(\alpha g_n^{-1}) = \mathrm{tr}_d(\alpha g_n) - (u^{-1} - 1)\mathrm{tr}_d(\alpha e_{d,n}) + (u^{-1} - 1)\mathrm{tr}_d(\alpha e_{d,n} g_n). \tag{17}$$

In order that the invariant agrees on the closures of the braids $\alpha{\sigma_n}^{-1}$ and $\alpha\sigma_n$ we need that $\mathrm{tr}_d(\alpha g_n^{-1})$ factorizes through $\mathrm{tr}_d(\alpha)$, just as $\mathrm{tr}_d(\alpha g_n)$ does. Indeed, for the first term we have: $\mathrm{tr}_d(\alpha g_n) = z\,\mathrm{tr}_d(\alpha)$. Further:

$$\mathrm{tr}_d(\alpha e_{d,n} g_n) = \frac{1}{d}\sum_{m=0}^{d-1} \mathrm{tr}_d(\alpha t_n^m t_{n+1}^{-m} g_n) = \frac{1}{d}\sum_{m=0}^{d-1} z\,\mathrm{tr}_d(\alpha) = z\,\mathrm{tr}_d(\alpha) \tag{18}$$

since $\mathrm{tr}_d(\alpha t_n^m t_{n+1}^{-m} g_n) = \mathrm{tr}_d(\alpha t_n^m g_n t_n^{-m}) = z\,\mathrm{tr}_d(\alpha t_n^m t_n^{-m}) = z\,\mathrm{tr}_d(\alpha)$. So, we need that $\mathrm{tr}_d(\alpha e_{d,n})$ also factorizes through $\mathrm{tr}_d(\alpha)$. Unfortunately, we do not have, in general, such a nice formula for $\mathrm{tr}_d(\alpha e_{d,n})$. The underlying reason on the framed braid level (which is the natural interpretation for elements in $\mathrm{Y}_{d,n}(u)$) is that $e_{d,n}$ involves the nth strand of the braid α. Yet, by imposing some conditions on the indeterminates x_i of the trace tr_d it is possible to have this factorization.

4.2. *The E–system.* Set $X_d = \{x_0, x_1, \ldots, x_{d-1}\}$ a set of d complex numbers. We shall say that X_d satisfies the *E–condition* if the x_i's are solutions of the following non–linear system of equations:

$$\begin{array}{rcl} \mathrm{E}_d^{(1)} & = & \mathrm{x}_1\mathrm{E}_d^{(0)} \\ \mathrm{E}_d^{(2)} & = & \mathrm{x}_2\mathrm{E}_d^{(0)} \\ & \vdots & \\ \mathrm{E}_d^{(d-1)} & = & \mathrm{x}_{d-1}\mathrm{E}_d^{(0)} \end{array} \tag{19}$$

where $\mathrm{E}_d^{(m)}$ is the polynomial in variables $\mathrm{x}_1, \ldots, \mathrm{x}_{d-1}$ defined as:

$$\mathrm{E}_d^{(m)} = \sum_{s=0}^{d-1} \mathrm{x}_{m+s}\mathrm{x}_{d-s} \tag{20}$$

where, by definition, $\mathrm{x}_0 = \mathrm{x}_d = 1$, and the sub-indices are regarded modulo d. We shall refer to the system above as the *(E, d)–system* or simply the *E–system*. For example, in the case $d = 3$ we have the E–system:

$$\begin{array}{rcl} \mathrm{x}_1 + \mathrm{x}_2^2 & = & 2\mathrm{x}_1^2\mathrm{x}_2 \\ \mathrm{x}_1^2 + \mathrm{x}_2 & = & 2\mathrm{x}_1\mathrm{x}_2^2 \end{array}$$

We then have the following result (compare with Theorem 6 in [4]).

Theorem 4. *If $X_{d,S}$ is a solution of the E–system then for all $\alpha \in \mathrm{Y}_{d,n}$ we have:*

$$\mathrm{tr}_d(\alpha e_{d,n}) = \mathrm{tr}_d(\alpha)\,\mathrm{tr}_d(e_{d,n}).$$

For the proof of Theorem 4 we need to consider all different cases for α being an element in the inductive basis of $\mathrm{Y}_{d,n}(u)$. See [4] for details.

We still need to establish, of course, that the set of solutions of the E–system is non–empty. For $a \in \mathbb{Z}/d\mathbb{Z}$ we denote $\exp_a$ the exponential character of the group $\mathbb{Z}/d\mathbb{Z}$, that is:

$$\exp_a(k) := \cos\frac{2\pi ak}{d} + i\sin\frac{2\pi ak}{d} \qquad (k \in \mathbb{Z}/d\mathbb{Z}).$$

Theorem 5. *The solutions of system (19) above are parametrized by the non–empty subsets S of $\mathbb{Z}/d\mathbb{Z}$. More precisely, a subset S defines the solution $X_{d,S} = \{x_0, x_1, \ldots, x_{d-1}\}$, where:*

$$x_k = \frac{1}{|S|}\sum_{s\in S}\exp_s(k) \qquad (0 \le k \le d-1).$$

Proof. See Appendix in [4]. □

Let $X_{d,S}$ be a solution of the E–system. A direct computation yields that the value of the tr_d on $e_{d,i}$ (with respect to $X_{d,S}$) is:

$$\mathrm{tr}_d\,(e_{d,i}) = \frac{1}{|S|} \qquad (1 \le i \le n-1). \tag{21}$$

For a thorough discussion and full proofs related to the E–condition and the E–system we refer the reader to [4].

4.3. For $d|d'$ we denote $\mathrm{s}^d_{d'}$ a section map of the natural epimorphism $\vartheta^{d'}_d$ of (5). By taking a section $\mathrm{s}^d_{d'}$ any solution of the (E,d)–system can be lifted trivially to a solution of the (E,d')–system. Indeed: If $X_{d,S}$ is a solution of the (E,d)–system, then $X_{d',S'}$ is a solution of the (E,d')–system, where $S' := \mathrm{s}^d_{d'}(S)$. A more interesting lifting can be constructed as follows. Define $S^d_{d'} = \{\mathrm{s}^d_{d'}(a) + b\,;a \in S,\, b \in \ker \vartheta^{d'}_d\}$. Then we define the lifting $X_{d,d',S}$ of $X_{d,S}$ as:

$$X_{d,d',S} := X_{d',S^d_{d'}} \qquad (S \subseteq \mathbb{Z}/d\mathbb{Z}) \tag{22}$$

Notice that $|X_{d,d',S}| = |S|d'/d$ and $X_{d,d,S} = X_{d,S}$.

Lemma 3. *For $d|d'|d''$ and S non–empty subset of $\mathbb{Z}/d\mathbb{Z}$ we have:*

$$X_{d,d'',S} = X_{d',d'',S'}$$

where $S' := S^{d_{d'}}$.

Proof. According to the definition of $X_{S,d}$ it is enough to prove that:

$$\left(S^d_{d'}\right)^{d'}_{d''} = S^d_{d'}$$

Now the elements in $\left(S^d_{d'}\right)^{d'}_{d''}$ are in the form $z := \mathrm{s}^{d'}_{d''}(x)+y$, where $x \in S^d_{d'}$ and $y \in \ker \vartheta^{d''}_{d'}$. The element x is in the form $x = \mathrm{s}^d_{d'}(\mu) + \nu$, where $\mu \in S$ and $\nu \in \ker \vartheta^{d'}_d$. So we can re–write z as:

$$z = \mathrm{s}^{d'}_{d''}\left(\mathrm{s}^d_{d'}(\mu) + \nu\right) + y = \mathrm{s}^{d'}_{d''}\left(\mathrm{s}^d_{d'}(\mu)\right) + \mathrm{s}^{d'}_{d''}(\nu) + y = \mathrm{s}^d_{d''}(\mu) + \mathrm{s}^{d'}_{d''}(\nu) + y$$

But $\mathrm{s}^{d'}_{d''}(\nu) + y$ belong to the $\ker \vartheta^{d''}_d$; hence $z \in S^d_{d'}$. Thus $\left(S^d_{d'}\right)^{d'}_{d''} = S^d_{d'}$. □

We showed that solutions of the E–system lift to solutions on the adelic level.

5. An adelic extension of the Jones polynomial

5.1. *Isotopy invariants from* tr_d. Given a solution $X_{d,S}$ of the E–system, Eq. 17 can be rewritten as follows, using Theorem 4:

$$\mathrm{tr}_d(\alpha g_n^{-1}) = \frac{z + (u-1)\zeta_{d,S}}{u}\,\mathrm{tr}_d(\alpha) \tag{23}$$

where, for all i:

$$\zeta_{d,S} := \mathrm{tr}_d(e_{d,i}) = \frac{1}{|S|}.$$

Let now $\mathcal{L}$ be the set of oriented links in S^3. Recall that by Alexander's theorem every link type may be represented by a closed braid. For the solution $X_{d,S}$ of the E–system we wish to define a link isotopy invariant $\Delta_{d,S}$. In order that $\Delta_{d,S}(\widehat{\alpha\sigma_n}) = \Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}})$, for $\alpha \in B_n$, we apply a re-scaling via the homomorphism:

$$\begin{array}{rccc} \delta: & B_n & \longrightarrow & \mathrm{Y}_{d,n} \\ & \sigma_i & \mapsto & \sqrt{\lambda}g_i \end{array} \tag{24}$$

where:

$$\lambda := \frac{z - (1-u)\zeta_{d,S}}{uz}$$

Finally, in order that $\Delta_{d,S}(\widehat{\alpha\sigma_n}) = \Delta_{d,S}(\widehat{\alpha})$ we need to do a normalization. So, we define the following map on the set $\mathcal{L}$.

Definition 3. Let $\alpha \in B_n$. We define the map $\Delta_{d,S}$ on the closure $\widehat{\alpha}$ of α as follows:

$$\Delta_{d,S}(\widehat{\alpha}) := \left(\frac{1-\lambda u}{\sqrt{\lambda}(1-u)\zeta_{d,S}} \right)^{n-1} (\mathrm{tr}_d \circ \delta)(\alpha)$$

Equivalently, setting

$$D := \frac{1-\lambda u}{\sqrt{\lambda}(1-u)\zeta_{d,S}}$$

we can write:

$$\Delta_{d,S}(\widehat{\alpha}) = D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)}\mathrm{tr}_d(\flat_{d,n}(\alpha))$$

where $\epsilon(\alpha)$ is the algebraic sum of the exponents of the σ_i's in the braid word α and where $\flat_{d,n}$ was defined in (11).

Theorem 6. *For the solution $X_{d,S}$ of the E–system, $\Delta_{d,S}$ is a 2–variable isotopy invariant for oriented links, depending on the variables u, z.*

Proof. We need to show that $\Delta_{d,S}$ is well–defined on isotopy classes of oriented links. According to Theorem 1, it suffices to prove that $\Delta_{d,S}$ is consistent with moves (i) and (ii). From the facts that $\epsilon(\alpha\alpha') = \epsilon(\alpha'\alpha)$ and $\mathrm{tr}_d(ab) = \mathrm{tr}_d(ba)$, it follows that $\Delta_{d,S}$ respects move (i). Let now $\alpha \in B_n$. Then $\alpha\sigma_n \in B_{n+1}$ and $\epsilon(\alpha\sigma_n) = \epsilon(\alpha) + 1$. Hence:

$$\Delta_{d,S}(\widehat{\alpha\sigma_n}) = D^n(\sqrt{\lambda})^{\epsilon(\alpha\sigma_n)}\,\mathrm{tr}_d(\flat_{d,n}(\alpha\sigma_n)) = D^n(\sqrt{\lambda})^{\epsilon(\alpha)+1}\,\mathrm{tr}_d(\flat_{d,n}(\alpha)g_n) = D\sqrt{\lambda}\,z\,\Delta_{d,S}(\widehat{\alpha})$$

where we used that $\mathrm{tr}_d(\flat_{d,n}(\alpha)g_n) = z\,\mathrm{tr}(\flat_{d,n}(\alpha))$. Now:

$$z = \frac{(1-u)\zeta_{d,S}}{1-\lambda u},$$

so:

$$D\sqrt{\lambda}\,z = 1.$$

Therefore, $\Delta_{d,S}(\widehat{\alpha\sigma_n}) = \Delta_{d,S}(\widehat{\alpha})$. Finally, we will prove that $\Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}}) = \Delta_{d,S}(\widehat{\alpha})$. Indeed:

$$\Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\alpha\sigma_n^{-1})}\mathrm{tr}_d(\flat_{d,n}(\alpha\sigma_n^{-1})) = D^n(\sqrt{\lambda})^{\epsilon(\alpha)-1}\,\mathrm{tr}_d(\flat_{d,n}(\alpha)g_n^{-1}).$$

Resolving g_n^{-1} from Eq. 16 we obtain:

$$\Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\alpha)-1}\,\left[z - (u^{-1}-1)\zeta_{d,S} + (u^{-1}-1)z\right]\,\mathrm{tr}_d(\flat_{d,n}(\alpha)).$$

Also, from Theorem 4 and Eq. 18 we have:

$$\mathrm{tr}_d(\flat_{d,n}(\alpha e_{d,n})) = \zeta_{d,S}\,\mathrm{tr}_d(\flat_{d,n}(\alpha)) \quad \text{and} \quad \mathrm{tr}_d(\flat_{d,n}(\alpha)e_{d,n}g_n) = z\,\mathrm{tr}_d(\flat_{d,n}(\alpha)).$$

Therefore:

$$\Delta_{d,S}(\widehat{\alpha\sigma_n^{-1}}) = D^n(\sqrt{\lambda})^{\epsilon(\alpha)-1}\,\frac{z+(u-1)\zeta_{d,S}}{u}\,\mathrm{tr}_d(\flat_{d,n}(\alpha)) = \frac{D}{\sqrt{\lambda}}\,\frac{z+(u-1)\zeta_{d,S}}{u}\,\Delta_{d,S}(\widehat{\alpha}) =$$

$\Delta_{d,S}(\widehat{\alpha})$. Hence the proof is concluded. □

We have defined an infinite family of 2–variable isotopy invariants for oriented classical links.

5.2. *Computations.* We shall first give some formulas that are useful for computations. For powers of g_i we can deduce by induction the following formulae.

Lemma 4. *Let $m \in \mathbb{Z}, k \in \mathbb{N}$. (i) For m positive, define $\alpha_m = (u-1)\sum_{l=0}^{k-1} u^{2l}$ if $m = 2k$ and $\beta_m = u(u-1)\sum_{l=0}^{k-1} u^{2l}$ if $m = 2k+1$. Then:*

$$g_i^m = \begin{cases} 1 + \alpha_m\, e_{d,i} - \alpha_m\, e_{d,i}\, g_i & \text{if } \ m = 2k \\ g_i - \beta_m\, e_{d,i} + \beta_m\, e_{d,i} g_i & \text{if } \ m = 2k+1 \end{cases}$$

(ii) For m negative, define $\alpha'_m = u^{-1}(u^{-1}-1)\sum_{l=0}^{k-1} u^{-2l}$ if $m = -2k$ and $\beta'_m = (u^{-1}-1)\sum_{l=0}^{k-1} u^{-2l}$ if $m = -2k+1$. Then:

$$g_i^m = \begin{cases} 1 + \alpha'_m\, e_{d,i} - \alpha'_m\, e_{d,i}\, g_i & \text{if } \ m = -2k \\ g_i - \beta'_m\, e_{d,i} + \beta'_m\, e_{d,i} g_i & \text{if } \ m = -2k+1 \end{cases}$$

We now proceed with some basic computations. Clearly, for the unknot O, $\Delta_{d,S}(\mathrm{O}) = 1$. For the Hopf link and Trefoil Knots we have:

• Let $\mathrm{H} = \widehat{\sigma_1^2}$, the Hopf link. We find $\mathrm{tr}_d(g_1^2) = 1 + (u+1)(\zeta_{d,S} - z)$ and $\epsilon(\sigma_1^2) = 2$. Then:

$$\Delta_{d,S}(\mathrm{H}) = \frac{1-\lambda u}{(1-u)\zeta_{d,S}}\sqrt{\lambda}\ (1 + (u+1)(\zeta_{d,S} - z)) = z^{-1}\sqrt{\lambda}\,(1 + (u+1)(\zeta_{d,S} - z))\,.$$

• Let $\mathrm{T} = \widehat{\sigma_1^3}$, the right-handed trefoil. From Lemma 4 we have $g_1^3 = g_1 - u(u-1)e_{d,1} + u(u-1)e_{d,1}g_1$. Hence: $\mathrm{tr}_d(g_1^3) = z - u(u-1)\zeta_{d,S} + u(u-1)z$. Moreover $\epsilon(\sigma_1^3) = 3$. Then, using that $1 - \lambda u = z^{-1}\zeta_{d,S}(1-u)$, we obtain:

$$\begin{aligned} \Delta_{d,S}(\mathrm{T}) &= D(\sqrt{\lambda})^3\,[(u(u-1)+1)z - u(u-1)\zeta_{d,S}] \\ &= \frac{\lambda}{z}\left[(u^2 - u + 1)z - (u^2 - u)\zeta_{d,S}\right]. \end{aligned}$$

• Let, finally, $-\mathrm{T} = \widehat{\sigma_1^{-3}}$, the left-handed trefoil. From Lemma 4 we have $g_1^{-3} = g_1 - (u^{-1}-1)(u^{-2}+1)e_{d,1} + (u^{-1}-1)(u^{-2}+1)e_{d,1}g_1$. Hence: $\mathrm{tr}_d(g_1^{-3}) = z - (u^{-1}-1)(u^{-2}+1)\zeta_{d,S} + (u^{-1}-1)(u^{-2}+1)z$. Moreover $\epsilon(\sigma_1^{-3}) = -3$. Then we obtain:

$$\Delta_{d,S}(-\mathrm{T}) \;=\; D(\sqrt{\lambda})^{-3}\left[(u^{-3} - u^{-2} + u^{-1})z - (u^{-3} - u^{-2} + u^{-1} - 1)\zeta_{d,S}\right],$$

where we recall that $D = \frac{1-\lambda u}{\sqrt{\lambda}(1-u)\zeta_{d,S}}$.

5.3. *A cubic skein relation for $\Delta_{d,S}$.* Let L_+, L_-, L_0 be diagrams of oriented links, which are all identical, except near one crossing, where they differ by the ways indicated in Figure 1. We shall try to establish a skein relation satisfied by the invariant $\Delta_{d,S}$. Indeed, by the Alexander theorem we may assume that L_+ is in braided form and that $L_+ = \widehat{\beta\sigma_i}$ for some $\beta \in B_n$. Also that $L_- = \widehat{\beta\sigma_i^{-1}}$ and that $L_0 = \widehat{\beta}$. Apply now relation (16) for the

g_i^{-1} in the expression below, noting that $\epsilon(\beta\sigma_i^{-1}) = \epsilon(\beta) - 1$ and $\epsilon(\beta\sigma_i) = \epsilon(\beta) + 1$:

$$\begin{aligned}\Delta_{d,S}(L_-) &= D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta\sigma_i^{-1})}\mathrm{tr}_d(\flat_{d,n}(\beta)g_i^{-1}) \\ &= D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta)-1}\big[\mathrm{tr}_d(\flat_{d,n}(\beta)g_i) - (u^{-1}-1)\,\mathrm{tr}_d(\flat_{d,n}(\beta)e_{d,i}) \\ &\quad +(u^{-1}-1)\,\mathrm{tr}_d(\flat_{d,n}(\beta)e_{d,i}\,g_i)\big] \\ &= \frac{1}{\lambda}\Delta_{d,S}(L_+) - D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta)-1}(u^{-1}-1)\,\mathrm{tr}_d(\flat_{d,n}(\beta)e_{d,i}) \\ &\quad +D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta)-1}(u^{-1}-1)\,\mathrm{tr}_d(\flat_{d,n}(\beta)e_{d,i}\,g_i).\end{aligned}$$

The problem is that the algebra words $\flat_{d,n}(\beta)e_{d,i}$ and $\flat_{d,n}(\beta)e_{d,i}\,g_i$ do not have a natural lifting in the braid groups, even if we break the $e_{d,i}$'s according to (3). This was not the case in [4], where we were dealing with framed braids and all algebra generators had natural liftings in the framed braid groups.

Yet, we have in the algebra $\mathrm{Y}_{d,n}$ the following 'closed' relation.

Lemma 5. *The generators g_i of the Yokonuma–Hecke algebra $\mathrm{Y}_{d,n}$ satisfy the cubic relation:*

$$g_i^3 = -ug_i^2 + g_i + u \tag{25}$$

Equivalently,

$$g_i^{-1} = u^{-1}g_i^2 + g_i - u^{-1} \tag{26}$$

Proof. From Lemma 4 we find the relation $g_i^3 = g_i+(u-1)e_{d,i}g_i-(u-1)e_{d,i}g_i^2$. Substituting (2) and replacing the expression $(u-1)e_{d,i} - (u-1)e_{d,i}g_i$ by the expression $g_i^2 - 1$ we arrive at the stated cubic relation. □

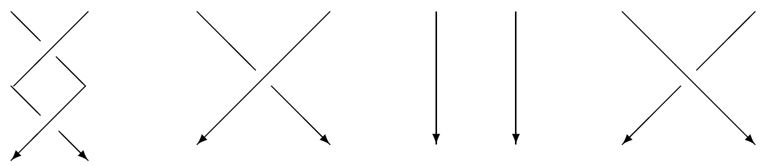

FIGURE 1. L_{++}, L_+, L_0 and L_-

We then have the following result.

Proposition 2. *The invariant $\Delta_{d,S}$ satisfies the following cubic skein relation:*

$$\sqrt{\lambda}\,\Delta_{d,S}(L_-) = \frac{1}{\lambda u}\,\Delta_{d,S}(L_{++}) + \frac{1}{\sqrt{\lambda}}\,\Delta_{d,S}(L_+) - \frac{1}{u}\,\Delta_{d,S}(L_0). \tag{27}$$

Proof. By the same reasoning as above we may assume that $L_0 = \widehat{\beta}$ for some $\beta \in B_n$. Also that $L_+ = \widehat{\beta\sigma_i}$, $L_{++} = \widehat{\beta\sigma_i^2}$ and $L_- = \widehat{\beta\sigma_i^{-1}}$. Apply now relation (26) from Lemma 5 in the expression below, noting that $\epsilon(\beta\sigma_i^{-1}) = \epsilon(\beta)-1$, $\epsilon(\beta\sigma_i) = \epsilon(\beta)+1$ and $\epsilon(\beta\sigma_i^2) = \epsilon(\beta)+2$.

$$\begin{aligned}
\Delta_{d,S}(L_-) &= D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta\sigma_i^{-1})}\mathrm{tr}_d(\flat_{d,n}(\beta)g_i^{-1}) \\
&= D^{n-1}(\sqrt{\lambda})^{\epsilon(\beta)-1}\left[u^{-1}\,\mathrm{tr}_d(\flat_{d,n}(\beta)g_i^2) + \mathrm{tr}_d(\flat_{d,n}(\beta)g_i) - u^{-1}\,\mathrm{tr}_d(\flat_{d,n}(\beta))\right] \\
&= \frac{1}{(\sqrt{\lambda})^3 u}\Delta_{d,S}(L_{++}) + \frac{1}{\lambda}\Delta_{d,S}(L_+) - \frac{1}{\sqrt{\lambda}u}\Delta_{d,S}(L_0).
\end{aligned}$$

□

5.4. *An isotopy invariant from* tr_∞. In this subsection we extend the values of the invariants $\Delta_{d,S}$ to the adelic context. By (13) the braid group B_n is represented in $\mathrm{Y}_{\infty,n} = \varprojlim_{d\in\mathbb{N}^\sim} \mathrm{Y}_{d,n}$ via the map $\flat_{\infty,n} = \varprojlim_{d\in\mathbb{N}^\sim} \flat_{d,n}$. Further, by Theorem 3, elements in $\mathrm{Y}_{\infty,n}$ map, via the Markov trace $\mathrm{tr}_{\infty,n} = \varprojlim_{d\in\mathbb{N}^\sim} \mathrm{tr}_{d,n}$, in the ring $\varprojlim_{d\in\mathbb{N}^\sim} R[X_d]$, where $R = \mathbb{C}[z]$.

For any $d|d'$, now, the connecting ring epimorphism $\xi_d^{d'}$ (recall (14)) yields a connecting epimorphism $\Xi_d^{d'}$ from the ring of rational functions $\mathbb{C}(z, X_{d'})$ to the ring of rational functions $\mathbb{C}(z, X_d)$.

Lemma 6. *The following diagram is commutative.*

$$\begin{tikzcd}
\mathcal{L} \arrow[r, "\Delta_{d',S}"] \arrow[d, "\mathrm{Id}"'] & \mathbb{C}\left(z, X_{d'}\right) \arrow[d, "\Xi_d^{d'}"] \\
\mathcal{L} \arrow[r, "\Delta_{d,S}"'] & \mathbb{C}\left(z, X_d\right)
\end{tikzcd} \tag{28}$$

We shall further denote by R_∞ the field of fractions of $\varprojlim_{d\in\mathbb{N}^\sim} R[\mathrm{X}_d]$. Taking now inverse limits in the diagram of Lemma 6 we obtain the map $\Delta_{\infty,S} := \varprojlim_{d\in\mathbb{N}^\sim} \Delta_{d,S}$ and we have the following.

Theorem 7. *If for all d the set X_d satisfies the E–condition, then the map*

$$\begin{array}{rcccl}
\Delta_{\infty,S} & : & \mathcal{L} & \longrightarrow & R_\infty \\
 & & \widehat{\alpha} & \mapsto & (\Delta_{d,S}(\widehat{\alpha}), \Delta_{d',S}(\widehat{\alpha}), \ldots)
\end{array}$$

for any $\alpha \in B_\infty$ is an isotopy invariant of oriented links in S^3. Moreover:

$$\Delta_{\infty,S}(\widehat{\alpha}) = \left(\frac{1-\lambda u}{\sqrt{\lambda}(1-u)\zeta_{d,S}}\right)^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)}\mathrm{tr}_\infty(\flat_{\infty,n}(\alpha)) = D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)}\mathrm{tr}_\infty(\flat_{\infty,n}(\alpha)).$$

Proof. By Lemma 3 we have non-trivial solutions of the E–system in the adelic context. Let now $\beta, \alpha \in B_\infty$ be Markov equivalent braids. Then, any isotopy invariants agrees on the closures $\widehat{\beta}$ and $\widehat{\alpha}$. So, $\Delta_{d,S}(\widehat{\beta}) = \Delta_{d,S}(\widehat{\alpha})$, $\Delta_{d',S}(\widehat{\beta}) = \Delta_{d',S}(\widehat{\alpha})$, etc. Hence: $\Delta_{\infty,S}(\widehat{\alpha}) = \Delta_{\infty,S}(\widehat{\beta})$. Moreover, we have:

$$\begin{aligned}
\Delta_{\infty,S}(\widehat{\alpha}) &= (\Delta_{d,S}(\widehat{\alpha}), \Delta_{d',S}(\widehat{\alpha}), \ldots) \\
&= (D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)}\mathrm{tr}_d(\flat_{d,n}(\alpha)), D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)}\mathrm{tr}_{d'}(\flat_{d',n}(\alpha)), \ldots) \\
&= D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)}(\mathrm{tr}_d(\flat_{d,n}(\alpha)), \mathrm{tr}_{d'}(\flat_{d',n}(\alpha)), \ldots) \\
&= D^{n-1}(\sqrt{\lambda})^{\epsilon(\alpha)}\mathrm{tr}_\infty(\flat_{\infty,n}(\alpha)).
\end{aligned}$$

□

The link invariant $\Delta_{\infty,S}$ is an adelic extension of the Jones polynomial.

References

[1] V.F.R. Jones, *Hecke algebra representations of braid groups and link polynomials*, Ann. Math. **126**, 335–388 (1987).

[2] J. Juyumaya, *Markov trace on the Yokonuma-Hecke algebra*, J. Knot Theory and its Ramifications **13**, 25–39 (2004).

[3] J. Juyumaya, S. Lambropoulou, *p-adic framed braids*, Topology and its Applications **154**, 1804–1826 (2007).

[4] J. Juyumaya, S. Lambropoulou, *p-adic framed braids II*, submitted for publication. See ArXiv:0905.3626v1 [math.GT] 22 May 2009.

[5] J. Juyumaya, S. Lambropoulou, *An invariant for singular knots*, to appear in J. Knot Theory and its Ramifications, **18**, No. 7 (2008) 1–16. See also ArXiv:0905.3665v1 [math.GT] 22 May 2009.

[6] S. Lambropoulou, *Knot theory related to generalized and cyclotomic Hecke algebras of type* B, J. Knot Theory and its Ramifications **8**, No. 5, 621–658 (1999).

[7] L. Ribes and P. Zalesskii, *Profinite Groups*, A Ser. Mod. Sur. Math. 40, Springer (2000).

[8] Wilson, *Profinite Groups*, London Math. Soc. Mono., New Series 19, Oxford Sc. Publ. (1998).

[9] T. Yokonuma, *Sur la structure des anneaux de Hecke d'un groupe de Chevallley fini,* C.R. Acad. Sc. Paris, **264**, 344–347 (1967).

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